THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4050 Real (Spring 2018)
Midterm Solution

1. For an increasing sequence of measurable sets $\{E_n\}$, if there exists n_0 s.t. $m(E_{n_0}) = \infty$, then by monotonicity of measure, the equality holds. If $m(E_n) < \infty$ for all $n \in \mathbb{N}$, define $A_1 = E_1$, $A_n = E_n \setminus E_{n-1}$ for $n \geq 2$. By monotonicity of E_n, $\{A_n\}$ are mutually disjoint and $\cup_{n=1}^{\infty} A_n = \cup_{n=1}^{\infty} E_n$. By countable additivity of m,

$$\begin{aligned} m(\cup_{n=1}^{\infty} E_n) =& m(\cup_{n=1}^{\infty} A_n) \\ &= m(E_1) + \sum_{n=2}^{\infty} [m(E_n) - m(E_{n-1})] \\ &= \lim_{n\to\infty} [m(E_n) - m(E_1)] + m(E_1), \end{aligned}$$

result follows.

For a decreasing sequence of measurable sets $\{E_n\}$ with $m(E_1) < \infty$, Let $D_n = E_1 \setminus E_n$, hence $\{D_n\}$ is an increasing sequence of measurable sets. It follows from above that

$$m(\cup_{n=1}^{\infty} D_n) = \lim_{n\to\infty} m(D_n).$$

Hence, note that $\cup_{n=1}^{\infty} D_n = E_1 \setminus \cap_{n=1}^{\infty} E_n$ and $m(D_n) = m(E_1) - m(E_n)$, we have

$$m(E_1 \setminus \cap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} [m(E_1) - m(E_n)].$$

By measurability of $\cap_{n=1}^{\infty} E_n$ and $m(E_1) < \infty$, result follows.

2. Let $c \in \mathbb{R}$,
$\{x \in E | \max\{f_1, f_2\}(x) > c\} = \{x \in E | f_1(x) > c\} \cup \{x \in E | f_2(x) > c\} \in \mathcal{M}$. Hence, $\max\{f_1, f_2\} \in \mathcal{MF}$.
In particular, for a sequence of measurable functions $\{h_n\}$, define $g = \sup_n h_n$, we have, by $\mathcal{M}$ being a σ-algebra,

$$\{x \in E | g(x) > c\} = \cup_{n=1}^{\infty} \{x \in E | h_n(x) > c\} \in \mathcal{M}.$$

i.e. $g \in \mathcal{MF}$. Similarly, we have $\inf h_n \in \mathcal{MF}$. Now, define $h_k = \sup_{n \geq k} f_n \in \mathcal{MF}$. Hence, $f = \lim_n f_n = \limsup f_n = \inf_k h_k \in \mathcal{MF}$.

3. For all $n \in \mathbb{N}$, define $E_n = \{x \in E : |g(x)| \geq n\}$. Note that, for all n,

$$m(\{x \in E : |g| = \infty\}) = m(\cap_k E_k) \leq m(E_n).$$

Since $g \in \mathcal{L}^1(E)$,

$$\infty > \int_E |g| \geq \int_{E_n} |g| \geq n m(E_n).$$

Result follows.
Now, we have f and g are finite a.e. Except on a null set N, $g - f$ is well defined, we can assign any value for

$g-f$ on N. Note that $|g-f| \leq |f|+|g|$ a.e., by monotonicity and linearity of Lebesgue integral, $g-f \in \mathcal{L}^1(E)$. By linearity, $\int_E (g-f) = \int_E g - \int_E f = 0$. $g-f \geq 0$ a.e., let $n \in \mathbb{N}$, let $K_n = \{x \in E | g-f \geq \frac{1}{n}\}$,

$$0 = \int_E (g-f) \geq \frac{1}{n} m(K_n) \geq 0$$

. i.e. $m(K_n) = 0$ for all n, hence,

$$m(\{x \in E : g-f > 0\}) = m(\cup_n K_n) \leq \sum_n m(K_n) = 0.$$

4. (i) Since $f \geq 0$, $\lambda(E) \geq 0$ for all $E \in \mathcal{M}$. Since $m(\phi) = 0$, we have $\lambda(\phi) = 0$. It remains to check countable additivity. Let $\{E_n\}$, mutually disjoint, be a sequence in (M). Note that for each $N \in \mathbb{N}$, $a_N := f\chi_{\cup_{n=1}^N E_n} \geq 0$. This sequence is monotone increasing in N, converges a.e. to $f\chi_{\cup_{n=1}^\infty E_n} \geq 0$ as $N \to \infty$. Hence, we can apply monotone convergence theorem,

$$\sum_{n=1}^{\infty} \lambda(E_n) = \lim_N \sum_n^N \lambda(E_n) = \lim_N \int_{\mathbb{R}} f\chi_{\cup_{n=1}^N E_n} = \int_{\mathbb{R}} f\chi_{\cup_{n=1}^\infty E_n} = \int_{\cup_{n=1}^\infty E_n} f = \lambda(\cup_{n=1}^\infty E_n)$$

(ii) For each $n \in \mathbb{N}$, define $h_n : \mathbb{R} \to \mathbb{R}$ by $h_n = \min\{f, n\}$. Check that, $h_n \geq 0$, monotonically increasing in n, converges a.e. to f as $n \to \infty$. By Monotone Convergence Theorem, we have

$$\lim_n \int_{\mathbb{R}} h_n = \int_{\mathbb{R}} f < \infty.$$

Hence, let $\epsilon > 0$, there exists N s.t.

$$\int_{\mathbb{R}} f - \int_{\mathbb{R}} h_N < \epsilon.$$

Let $0 < \delta < \frac{\epsilon}{2N}$, if $A \in \mathcal{M}$ with $m(A) < \delta$, by linearity and monotonicity,

$$\lambda(A) = \int_A f = \int_A f - \int_A h_N + \int_A h_N \leq \int_{\mathbb{R}} (f - h_N) + \epsilon/2 < \epsilon.$$

5. $A := \{x \in E | \alpha > D_- f(x) = \sup_{\delta > 0} \inf_{0 < x-y < \delta, y \in [a,b]} \dfrac{f(y) - f(x)}{y - x}\}$. This is well defined since $E \subset (a,b)$. By definition of sup and inf, let $x \in A$, let $\delta > 0$, there exists $d_{\delta,x} \in (0, \delta)$ small enough such that $\dfrac{f(x - d_{\delta,x}) - f(x)}{-d_{\delta,x}} < \alpha$ and $(x - \delta_x, x] \subset G$. Then, let $\mathcal{C} = \{(x - \delta_x, x]), \delta > 0, z \in A\}$. Result follows.